ON GAMES OF SURVIVAL

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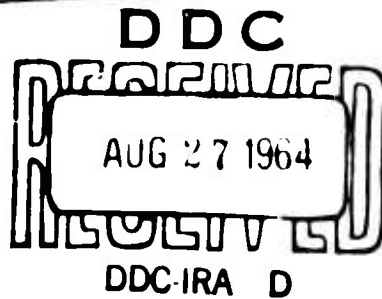
SUMMARY

In a game of survival, two players with limited resources play a zero-sum game repeatedly until one of them is ruined. The solution of the survival game gives one a measure of the value of resources in terms of survival probabilities. In this paper the zero-sum game is expressed as a finite matrix, but with (possibly) incommensurable entries; hence the number of different distributions of resources that can occur during a single play may be infinite. The existence of a value and optimal strategies is proved, using the theory of semi-martingales. A simple approximation to the solution is described, and several examples are discussed. ( )

ON GAMES OF SURVIVAL

Two gamblers, with limited resources of money, agree to play and replay the same zero—sum game until one of them is ruined. The "game of survival" that results is similar in many respects to the classic "gambler's ruin" problem, but there is one important difference: since the transition probabilities are controlled by the participants, and not by chance, there may be a positive probability of infinite repetition, with neither gambler being ruined. Thus, to save oneself and to destroy one's opponent are somewhat different objectives; in fact, the optimal strategies and the corresponding probabilities of ruin or survival will sometimes be found to depend on the value assigned to the case of double survival.

In this paper we propose to investigate thoroughly those games of survival where the underlying "money" game is given by an arbitrary, finite matrix of real numbers. The existence of solutions, and the extent to which they depend on the double—survival payoff, are the central topics. Our approach combines an analysis of certain game—theoretic functional equations with the theory of semimartingales. A number of examples, and methods of constructing and approximating the solutions, are also dis—cussed.

Previous writings on the subject include those of Bellman and LaSalle [3], Hausner [10], Peisakoff [11], and Bellman [1,2];

however, only certain special cases have so far been examined rigorously. Usually, it has been assumed that the number of accessible "states" (possible distributions of money during the play) is finite. It should be remarked that under such restrictions survival games become "recursive games" in the sense of Everett and the existence of a solution (assuming a constant double—survival payoff) becomes a simple corollary of his result [6]. Mention should also be made of the multi—dimensional survival games treated by Scarf [12], and the somewhat similar multi—component attrition games of Blackwell [4].

1. GENERAL INTRODUCTION AND EXAMPLES

Let $||a_{ij}||$ denote the matrix of the "money" game, let R be the sum of the resources of the two players, and let r_0 be the first player's initial fortune. Then, if player I chooses i_k on the kth round and player II chooses j_k, the new level of player I's fortune is given by:

(1) $$r_k = r_{k-1} + a_{i_k j_k}, \qquad k = 1, 2, \ldots;$$

a formula valid so long as $0 < r_{k-1} < R$. For r_{k-1} outside this interval we define $r_k = r_{k-1}$, serving the formal purpose of associating an infinite sequence $\{r_k\}$ with every play of the game, whether it terminates or not.

If one of the players is eventually ruined, the "utility" payoff to player I can be defined:

(2) $$P(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ 1 & \text{if } r \geq R, \end{cases}$$

where $r = \lim r_k$. If both players survive indefinitely, the payoff will be a number Q, which may be a function of the course of play. When we are not considering special cases we shall let Q be entirely arbitrary, assuming only $0 \leq Q \leq 1$. The pay-off to player II is taken to be 1 minus the payoff to player I.

Thus, the survival game is completely specified by the five elements: $||a_{ij}||$, P, Q, R, and r_0.

Assume for the moment that Q is a constant, and that the value of the game exists for every initial state r_0. Then it is easily proved that the value is a monotonic increasing function of r_0, and that it satisfies the functional equation:

$$(3) \qquad \phi(r) = \text{val } ||\phi(r+a_{ij})||, \qquad 0 < r < R,$$

with boundary conditions:

$$(4) \qquad \phi(r) = P(r) \qquad\qquad r \leq 0, \; r \geq R.$$

Here "val" denotes the ordinary minimax value of a matrix game.

Even if Q is not constant, equations (3) and (4) play a very fundamental role in the analysis. As we shall see in section 2 of this paper, there always exists at least one monotonic solution to (3), (4). If this solution is unique, then the value of the survival games exists and is independent of Q. If the solution is not unique, then the value may not exist, and it is not independent of Q if it does exist.

[To illustrate: in the first example below, all monotonic functions, and some others, are solutions of (3). In the second example, any linear or near-linear

function is a solution. In both cases the dependence
of the game on Q is intuitively obvious, since both
players have powerful "defensive" strategies that
prevent any action from taking place except on favor-
able ground. In the third

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 1 & -2 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Example 1   Example 2   Example 3

example equation (3) becomes trivial and irrelevant. The
game turns entirely on the properties of Q. An example
of Gale and Stewart [8] shows that the value does not
exist for certain payoffs of the form $Q = Q(i_1, j_2, i_3, \ldots)$.
Whether such indeterminacy can ever occur when Q has the
form $Q(r_1, r_2, \ldots)$ is an open question.]

A mixed strategy in the survival game can be represented
as a probability distribution on i (or j) for each round, as a
function of the past; this is the so-called "behavior strategy"
form. We shall call a mixed strategy locally optimal if for
every k the probabilities it prescribes for i_k (or j_k) are
optimal in the matrix game $||v(r_{k-1} + a_{ij})||$, v being the value
function of the survival game. Locally optimal strategies exist
whenever the value function exists, but they need not be optimal,
nor are optimal strategies necessarily locally optimal.

[Thus, in example 4, consider the strategy that picks $i_k = 2$ if and only if $r_{k-1} > 1$. It is clearly locally optimal, since the value function is identically 1, but it is not optimal if $R > 2$ and $Q < 1$. Again, in example 5, the strategy that always chooses $i_k = 3$ is locally optimal but not optimal if $Q < 1$ and $r_0 > 1$. (In this example the value depends on Q.) In example 6, the mixed

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Example 4   Example 5   Example 6

strategy that prescribes the probabilities $(1/3, 1/3, 1/3)$ for i_k if $j_{k-1} = 3$ and the probabilities $(1/2, 1/2, 0)$ if $j_{k-1} = 1$ or 2, or if $k = 1$, is optimal for player I, but it is not locally optimal, since it fails to take full advantage of the occasions when player II makes the "mistake" of playing $j = 3$.]

A semimartingale may be defined as a sequence of random variables $\{x_k\}$ such that the conditional expectation of each term is greater than or equal to the preceding term, thus:

$$E\left\{x_k \mid x_{k-1}, \ldots, x_0\right\} \geq x_{k-1}.$$

A fundamental theorem ([5], page 324) implies that a bounded semi-martingale converges with probability 1, and that its

limit x_∞ satisfies

$$E\left\{x_\infty \mid x_0\right\} \geq x_0 \ .$$

For our purposes, "bounded" can be taken to mean that the x_k themselves are bounded, uniformly in k, although the results stated are valid under much weaker conditions.

Let ϕ be any bounded solution of (3). We define a local ϕ-strategy to be a mixed strategy that always prescribes optimal probabilities for the games $||\phi(r_{k-1} + a_{ij})||$. Thus, in this terminology, a locally optimal strategy is a local v-strategy. If player I uses a local ϕ-strategy against an arbitrary strategy of player II, then the sequence $\left\{\phi(r_k)\right\}$ that is generated is a bounded semimartingale. (Note that $E\left\{\phi(r_k) \mid r_{k-1}, \ldots, r_0\right\} \geq \phi(r_{k-1})$ implies $E\left\{\phi(r_k) \mid \phi(r_{k-1}), \ldots, \phi(r_0)\right\} \geq \phi(r_{k-1})$, even though ϕ may not be one-one.) Hence we have convergence with probability 1, and

$$E\left\{\lim_{k\to\infty} \phi(r_k) \mid r_0\right\} \geq \phi(r_0) \ .$$

Now if ϕ satisfies (4) as well, the left side of this inequality can be expressed as

$$0 \cdot \text{prob}\left\{\text{I is ruined}\right\} + 1 \cdot \text{prob}\left\{\text{II is ruined}\right\} + \Theta \cdot \text{prob}\left\{\text{both survive}\right\},$$

where Θ is some number between 0 and 1. Hence:

(5) prob $\left\{\text{II is ruined}\right\} \geq \phi(r_0) - \Theta \cdot \text{prob}\left\{\text{both survive}\right\}$;

(6) prob $\left\{\text{I survives}\right\} \geq \phi(r_0) + (1-\Theta) \cdot \text{prob}\left\{\text{both survive}\right\}$.

Thus, such a strategy for player I guarantees that he will survive with probability $\geq \phi(r_0)$. If we could show that double survival has probability zero, at least for some particular local ϕ-strategy

of player I, then it would follow that he can guarantee himself
an expected payoff of $\phi(r_0)$, or more, regardless of the other's
strategy, and regardless of Q. A similar argument for player II
would then establish the existence of a value and optimal strategies
for the survival game, independent of Q.

In attempting to carry out a proof on the above lines, one
might hope to start with an arbitrary local ϕ-strategy and (i)
use the known convergence of $\{\phi(r_k)\}$ to establish convergence
of $\{r_k\}$; then (ii) use the convergence of $\{r_k\}$ to show that the
game must end; all with probability 1. Unfortunately, neither
(i) nor (ii) is unconditionally valid. In section 3 we proceed
by way of $\underline{\text{strictly}}$ monotonic approximants, for which (i) is
valid, and obtain thereby the existence of the value. In section
4 we obtain the existence of optimal strategies by working with
a special class of "interior ϕ-strategies," which make $\{r_k\}$ con-
verge even when ϕ is not strictly monotonic. However, in both
proofs it is necessary to assume that none of the a_{ij} is zero,
in order to make convergence of $\{r_k\}$ equivalent to termination
of play (step (ii)).

In section 5 we drop the zero-free condition on $||a_{ij}||$,
and find that a value still exists if Q is sufficiently regular.
However, the value may depend on Q (see examples 1, 2, 5 above),
and the players may not have optimal strategies (example 7 below).
Our proof parallels the one in section 3 (strictly monotonic
approximants), but is based on a more complicated functional
equation, to be discussed there.

Finally, in section 6 we will derive some estimates for the
value function that have much in common with the well-known ap-
proximate solutions of the classic "gambler's ruin" problem.
They have simple analytic forms, in contrast to the sharply
discontinuous nature of the exact value functions (see examples
8 and 9 below). The estimates become more precise if R is made
large compared to the a_{ij}, and they give exact information if
the a_{ij} are all ± 1, or ± 1 and 0. They also provide strategies
that are approximately optimal.

It should be noted that sections 3, 4, 5, and 6 are essen-
tially independent of one another.

[In example 7, player I can win with probability
approaching 1 if he always chooses i_k according to the
distribution $(1-\epsilon,\ \epsilon-\epsilon^2,\ \epsilon^2)$, with ϵ small but positive.
However, if $Q < 1$ he has no strictly optimal strategy.
Example 8 illustrates in a simple way some of the pos-
sibilities for the value function $v(r)$. Under optimal
play the first player's fortune describes a random walk

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & -1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix} \qquad \begin{pmatrix} 2+\epsilon_{11} & -2-\epsilon_{12} \\ -2-\epsilon_{21} & 2+\epsilon_{22} \end{pmatrix}$$

$\underline{\text{Example 7}} \qquad\qquad \underline{\text{Example 8}} \qquad\qquad \underline{\text{Example 9}}$

on $(0,\ R)$ with $+1$ and $-a$ having equal probability. The
value is just the probability of absorption at R. If
a is rational then the value is a finite step-function,
which can be determined exactly by solving a certain
system of linear equations. But if a is irrational
(with $R > 1+a > 1$), then the value function is

discontinuous on a set of points everywhere dense in (0, R);
it is strictly monotonic; and its derivative is almost every-
where 0. In example 9 the ϵ_{ij} are meant to be small pos-
itive incomensurables. We no longer have a simple random
walk as above, but it can be shown that for R = 3 the value
is constant in an interval slightly larger than (1, 2) and
has discontinuities everywhere dense in the rest of (0, R).
Whether the derivative vanishes almost everywhere in this
case is an open question.]

2. SOLUTIONS OF THE FUNCTIONAL EQUATIONS

A monotonic solution to (3), (4) can be constructed by an
iterative procedure. Define ϕ_0 by:

$$\phi_0(r) = \begin{cases} 0 & \text{if } r < R \\ 1 & \text{if } r \geq R \end{cases}$$

and let $\phi_n = T^n \phi_0$, where the transformation T is given by:

$$(7) \qquad T\phi(r) = \begin{cases} \text{val } ||\phi(r+a_{ij})|| & 0 < r < R \\ \phi(r) & r \leq 0, \ r \geq R. \end{cases}$$

It is clear that ϕ_n can be interpreted as the value function of
the finite, truncated game in which player I loses unless he
succeeds in ruining his opponent in n moves or less.

LEMMA 1. The sequence $\{\phi_n\}$ just defined converges
pointwise to a monotonic solution of (3), (4).

Proof. By construction, the fixed points of (7) are sol-
utions of (3), and conversely. Since T is continuous, it suffices
to show that $\lim \phi_n$ exists and is monotonic. This is accomplished
by showing inductively that $\phi_n(r)$ is monotonic increasing in
both n and r. The details present no difficulty whatever,
(Compare the much harder proof of lemmas 5 below.)

Let v_0 denote the limit of the ϕ_n, and let v_1 denote the
limit of the similar (descending) sequence $\left\{T^n \phi_0'\right\}$, beginning
with the function

$$\phi_0'(r) = \begin{cases} 0 & \text{if } r \leq 0 \\ 1 & \text{if } r > 0. \end{cases}$$

THEOREM 1. If Q ⫫ 0 then the value of the survival
game exists and is equal to $v_0(r_0)$. If Q ⫤ 1 then the
value exists and is equal to $v_1(r_0)$.

Proof. Player I can guarantee that

$$\text{prob} \left\{II \text{ is ruined}\right\} \geq \phi_n(r_0)$$

by following an optimal strategy for the nth truncated game (and
playing arbitrarily after the nth move). On the other hand,
player II can guarantee that

$$\text{prob} \left\{\text{I survives}\right\} \geq 1 - v_0(r_0)$$

by adopting a local v_0-strategy (see (6) above). But the payoff
of the Q ⫤ 0 game depends solely on whether player II survives
or not. Therefore $v_0(r_0)$ is its value. The other case is similar.

Note that the proof provides an optimal strategy for player II, but not player I, if $Q = 0$. The existence of this optimal strategy, and of the value, could have been deduced from the lower semi-continuity of the payoff, as a function of the pure strategies (see [9]). A similar remark applies to the $Q = 1$ game.

THEOREM 2. If (3), (4) have a unique solution ϕ, then the value of the survival games exists and is equal to $\phi(r_0)$, independently of Q.

Proof. As before, player I can ensure that
$$\text{prob} \{ \text{II is ruined} \} \geq \phi_n(r_0) .$$
Similarly player II can ensure that
$$\text{prob} \{ \text{I is ruined} \} \geq 1 - \phi_n'(r_0) .$$
But $\lim \phi_n = v_0 = \phi = v_1 = \lim \phi_n'$; hence $\phi(r_0)$ is the value of the game.

Note that this time we do not obtain an optimal strategy for either player.

The next lemma identifies v_0 and v_1 as the "extreme" solutions of (3), (4); and incidentally establishes a converse to theorem 2.

LEMMA 2. If ϕ is any solution of (3), (4) that is bounded between 0 and 1, then $v_0 \leq \phi \leq v_1$.

Proof. We observe that $\phi_0 \leq \phi$, and that $\phi_n \leq \phi$ implies $T\phi_n \leq \phi$. Hence $v_0 \leq \phi$. Symmetrically, $v_1 \geq \phi$.

COROLLARY. If the value function of the survival game exists and is independent of λ, then it is the only solution of (3), (4) bounded between 0 and 1.

The last provision is necessary, since "spurious" unbounded solutions do sometimes occur.

The next lemma shows that v_0 and v_1 usually have jumps at 0 and R, and characterizes the exceptions in terms of the matrix $||a_{ij}||$.

LEMMA 3. (A) The following are equivalent:

(i) $v_1(r)$ is continuous at $r = R$;

(ii) $v_1(r) = 1$ for $0 < r < R$;

(iii) $||a_{ij}||$ has a nonnegative row.

(B) The following are equivalent:

(i) $v_0(r)$ is continuous at $r = R$;

(ii) $v_0(r) = 1$ for $0 < r < R$;

(iii) every set of columns of $||a_{ij}||$, considered as a submatrix of $||a_{ij}||$, has a nonnegative row, not all zero.

Corresponding statements hold concerning continuity of v_0 and v_1 at $r = 0$.

Proof. (A) Obviously (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i). If (iii) is false there is a negative entry in each row. A strategy of playing all columns with equal probability, on every move, gives player II a probability $\geq n^{[-R/a]}$ of winning, if n is the number of columns and a is the smallest nonzero $|a_{ij}|$. This gives a

positive lower bound for $1 - v_1(r_0)$, independent of r_0, and makes v_1 discontinuous at R. Hence (i) => (iii).

(B) Obviously (ii) => (i). If (iii) is false there is a set of s columns on which player II can distribute his choices with equal probabilities $1/s$, giving him a probability $\geq s^{[-R/a]}$ of surviving. Hence $v_0(r_0)$ is bounded away from 1 and v_0 is discontinuous at R. Hence (i) => (iii). To complete the proof, suppose that (iii) holds but not (ii). Choose $r^* > 0$ so that $v_0(r^*) < v_0(r^*+a)$. Then $v_0(r^*)$ is strictly less than $v_0(r^*+a_{ij})$ whenever a_{ij} is positive. Let η be an optimal mixed strategy for II in the matrix game $||v_0(r^*+a_{ij})||$; let S be the set of columns j with $\eta_j > 0$; and let i_S be the nonnegative subrow, not all zero, whose existence is asserted by (iii). Then $v_0(r^*) \leq v_0(r^*+a_{i_S j})$ holds for j in S, with strict inequality at least once, and

$$\eta_j \, v_0(r^*) \leq \eta_j \, v_0(r^*+a_{i_S j})$$

holds for <u>all</u> j, with strict inequality at least once. Summing over j, and recalling the optimality of η, we obtain:

$$v_0(r^*) < \sum_j \eta_j v_0(r^*+a_{i_S j}) \leq \text{val} \, ||v_0(r^*+a_{ij})||.$$

But v_0 is a solution of (3), making the first and last terms equal. This contradiction establishes (iii) => (ii).

CORROLARY. If

$$\max_i \min_j a_{ij} < 0 < \min_j \max_i a_{ij}$$

then every bounded solution of (3), (4) has jumps at
0 and R.

It may be of interest to describe some near-optimal strategies
for player I in the event that (i), (ii), (iii) of (B) hold.
(Compare example 7 above.) Let S_0 be the set of all columns;
let i_0 be a row nonnegative and not identically zero on S_0;
let S_1 be the subset of S_0 on which $a_{i_0 j} = 0$; and so on. Then
we have $S_0 \supset \ldots \supset S_p \supset S_{p+1} = \phi$, for some p (proper inclusion
all the way), and moreover the $i_0, \ldots, i_p$ are all distinct.
Then it is easy to show that the probabilities:

$$\xi_{i_0} = 1 - \epsilon, \quad \xi_{i_1} = \epsilon - \epsilon^2, \quad \ldots, \quad \xi_{i_{p-1}} = \epsilon^{p-1} - \epsilon^p, \quad \xi_{i_p} = \epsilon^p,$$

all other $\xi_i = 0$, if used repeatedly by player I, guarantee with
probability $\geq 1 - \epsilon$ that the first nonzero a_{ij} to occur will be
positive. Hence player I wins with probability $\geq (1-\epsilon)^{-[-(R-r_0)/a]}$,
no matter what player II does. This bound goes to 1 as $\epsilon \to 0$.

3. EXISTENCE OF A VALUE WHEN $||a_{ij}||$ IS ZERO-FREE

This section will be devoted to the proof of the following
theorem:

THEOREM 3. If $||a_{ij}||$ is zero-free, then the
value of the survival game exists and is independent
of the payoff Q assigned to nonterminating play.

During the proof we shall work with certain generalized
survival games, having more general, bounded payoff functions

$P*(r)$. in place of the $P(r)$ of (2). The other elements of the game, namely $||a_{ij}||$, Q, R and r_0, remain as before. The functional equation (3) is still applicable, but with new boundary conditions:

(4*) $\phi(r) = P*(r)$ $r \leq 0,\ r \geq R.$

LEMMA 4. Suppose that (3), (4*) have a strictly monotonic solution $\phi*$. Then, if $||a_{ij}||$ is zero-free, the value of the generalized survival game exists and is equal to $\phi*(r_0)$.

Proof. Let player I use a local $\phi*$-strategy and player II an arbitrary strategy. Then $\{\phi*(r_k)\}$ is a bounded semimartingale, and converges with probability 1. Because $\phi*$ is strictly monotonic, $\{r_k\}$ also converges. This means that play terminates, since, with none of the $a_{ij} = 0$, the limit of $\{r_k\}$ must be outside (0, R). Hence

$$E\{P*(\lim r_k)\} = E\{\phi*(\lim r_k)\} \geq \phi*(r_0).$$

A similar argument for player II completes the proof.

We shall consider functions P* of the following form only (until section 6):

$$P*(r) = \begin{cases} \varepsilon(r-R-A) & \text{if } r \leq 0 \\ 1 + \varepsilon(r-R-A) & \text{if } r \geq R, \end{cases}$$

where $A = \max|a_{ij}|$, and ε is a positive constant. As $\varepsilon \to 0$ these functions approach $P(r)$ from below. Imitating the construction of section 2, we define $\phi*_n = T^n \phi*_0$ where T is the same transformation (7), and $\phi*_0$ is given by:

$$\phi*_0(r) = \begin{cases} \varepsilon(r-R-A) & \text{if } r \leq R \\ 1 + \varepsilon(r-R-A) & \text{if } r \geq R. \end{cases}$$

If it exists, $\lim \phi*_n$ is obviously a solution of (3), (4*).

LEMMA 5. If $\text{val}||a_{1j}|| \geq 0$, and if v_0 is not continuous at R, then for sufficiently small ϵ the sequence $\{\phi_n^*\}$ just defined converges pointwise to a strictly monotonic solution of (3), $(4*)$.

Proof. To show that the limit exists we observe first that

$$\phi_1^*(r) = \text{val}||\phi_0^*(r+a_{1j})|| \geq \text{val}||\epsilon(r-R-A+a_{1j})||$$

$$= \epsilon(r-R-A) + \epsilon \cdot \text{val}||a_{1j}|| \geq \epsilon(r-R-A)$$

$$= \phi_0^*(r)$$

if $0 < r < R$, and $\phi_1^*(r) = \phi_0^*(r)$ otherwise. Moreover, $\phi_n^* \geq \phi_{n-1}^*$ implies $\phi_{n+1}^* = T\phi_n^* \geq T\phi_{n-1}^* = \phi_n^*$. Since the sequence is obviously bounded, it therefore converges. To show that the limit is strictly monotonic we shall prove inductively that for ϵ sufficiently small the function $\phi_n^*(r) - \epsilon r$ is monotonic in r for each n. This is trivial for ϕ_0^*; assume it for ϕ_{n-1}^*. Case 1: Take $0 < r < s < R$. Then we have:

$$\phi_n^*(r) - \epsilon r = \text{val}\,||\phi_{n-1}^*(r+a_{1j})-\epsilon r||$$

$$\leq \text{val}\,||\phi_{n-1}^*(s+a_{1j})-\epsilon s||$$

$$= \phi_n^*(s) - \epsilon s.$$

Case 2: Take $r \leq 0 < s < R$. Then we have:

$$\phi_n^*(r) - \epsilon r = \phi_{n-1}^*(r) - \epsilon r$$

$$\leq \phi_{n-1}^*(s) - \epsilon s$$

$$\leq \phi_n^*(s) - \epsilon s$$

(using the first part of this proof in the last step).

Case 3: Take $0 < r < R \leq s$. Note that $\phi_0^* \leq v_0$ over the entire range of interest $(-A, R+A)$, and hence, by induction, that $\phi_n^* \leq v_0$. Since the latter is assumed discontinuous at R, we can select ε so that $\varepsilon A \leq 1 - v_0(R-)$. Then we have:

$$\phi_n^*(r) - \varepsilon r \leq \phi_n^*(R-) - \varepsilon R$$
$$\leq v_0(R-) - \varepsilon R$$
$$\leq 1 - \varepsilon A - \varepsilon R$$
$$= \phi_n^*(s) - \varepsilon s$$

(using case 1 in the first step). The other cases, namely $r < s \leq 0$, $r \leq 0 < R \leq s$, and $R \leq r < s$, are trivial. This completes the proof of lemma 5.

Proof of theorem 3. There is no loss of generality in assuming that $\mathrm{val}||a_{ij}|| \geq 0$. The theorem is trivial if $v_0 = 1$ in $(0, R)$; we may therefore assume (lemma 3) that v_0 is discontinuous at R. Lemmas 4 and 5 now give us well-defined value functions ϕ^* for our class of generalized games, for ε sufficiently small. As $\varepsilon \to 0$ their payoffs P^* converge uniformly to P, our original payoff function. It follows that $\lim_{\varepsilon \to 0} \phi^*(r_0)$ exists and is the value of the original game. This number is obviously independent of Q. This completes the proof.

As yet we know nothing about the existence of optimal strategies, unless $Q = 0$ or $Q = 1$. As example 4 showed, local v-strategies need not be optimal. However, the local ϕ^*-strategies, for given ε, are nearly optimal: it can be shown that they guarantee an expected payoff within $\varepsilon(R+2A)$ of the true value.

COROLLARY. If $||a_{ij}||$ is zero-free then (3),
(4) have a unique solution bounded between 0 and 1.

Proof. Theorem 3 and the corollary to lemma 2.

4. EXISTENCE OF OPTIMAL STRATEGIES

In this section we shall find optimal strategies for both players, under the assumption that $||a_{ij}||$ is zero-free. The strategies are locally optimal, with the added property that they force play to terminate with probability one. Their optimality is therefore independent of Q.

Let us first review some properties of matrix games. A pure strategy is said to be admissible if it appears with positive probability in at least one optimal strategy. We call an optimal strategy interior if it lies in the relative interior of the convex set of all optimal mixed strategies. Every matrix game has at least one interior optimal strategy for each player. For later reference we state two elementary facts:

(A)   If $val||b_{ij}|| = val||c_{ij}||$, with $b_{ij} \leq c_{ij}$, all i, j, and if $i*$ and $j*$ are admissible in $||b_{ij}||$ and $||c_{ij}||$ respectively, then $b_{i*j*} = c_{i*j*}$.

(B)   An interior optimal strategy "punishes" every inadmissible strategy of the other player; that is, if ξ is interior optimal and $j*$ is inadmissible in $||b_{ij}||$, then

$$\sum_i \xi_i b_{ij*} > val||b_{ij}||.$$

Returning to the survival game, we define a special class
of strategies for player I. Let ϕ be a monotonic solution of
(3), (4). For each number c for which the set $C = \phi^{-1}(c)$ is
not empty, let $\zeta(c)$ be an interior optimal strategy for player
I in the matrix game

$$(8) \qquad || \inf_{s \in C} \phi(s + a_{1j}) ||.$$

Define a mixed strategy for the survival game by the rule:
choose i_k according to the probability distribution $\zeta(\phi(r_{k-1}))$.
Such a strategy will be called an interior ϕ-strategy of player
I. Note that the same probabilities must be used each time the
same value of $\phi(r)$ comes up. Interior ϕ-strategies of player II
are defined similarly, with "sup" instead of "inf." We shall
see presently that an interior ϕ-strategy is also a local
ϕ-strategy. First we state our main result.

THEOREM 4. If ϕ is any monotonic solution of
(3), (4), and if $||a_{1j}||$ is zero-free, then the in-
terior ϕ-strategies are optimal, independently of Q,
and the value of the game is $\phi(r_0)$.

As stated, theorem 4 is independent of and includes theorem
3, and this independence will be maintained throughout the proof;
which takes up the rest of this section. Consequently we have
a separate proof of the existence of a value in the zero-free
case. Of course, ϕ is actually unique and equal to v, and our
real object is only to show that the interior v-strategies are
optimal.

LEMMA 6. An interior ϕ-strategy is also a local ϕ-strategy.

Proof. We must show that $\xi(\phi(r))$ is optimal in $||\phi(r+a_{ij})||$, for all r in (0, R). But $||\phi(r+a_{ij})||$ majorizes (8), and both matrices have the same value $\phi(r)$, by (3) and the fact that ϕ was assumed monotonic. Since $\xi(\phi(r))$ is optimal for (8) by definition, it is also optimal for $||\phi(r+a_{ij})||$.

LEMMA 7. Let $\phi(r) > 0$ and let r*, i*, j* be such that

(1) $r^* = r + a_{i*j*} < r,$

(ii) j* is admissible in $||\phi(r+a_{ij})||$,

(iii) $\xi_{i*}(\phi(r)) > 0.$

Then $\phi(r^*) \neq \phi(r).$

Proof. Suppose to the contrary that $\phi(r^*) = \phi(r)$. Then i* is admissible in $||\phi(r^*+a_{ij})||$ and the conditions of proposition (A) above are met, with regard to that matrix and $||\phi(r+a_{ij})||$. Hence $\phi(r^*+a_{i*j*}) = \phi(r+a_{i*j*})$, or $\phi(r+2a_{i*j*}) = \phi(r)$. Repeating this argument gives us $\phi(r') = \phi(r)$ for a sequence of r' that eventually becomes negative (since a_{i*j*} is negative), contradicting the hypothesis that $\phi(r) > 0$.

LEMMA 8. If player I uses an interior ϕ-strategy against any strategy of player II, then every possible play of the survival game has the property that, for each $k > 0$, one of the following is true:

(a) $r_{k-1} \leq 0$ or $r_{k-1} \geq R$;

(b) $\phi(r_{k-1}) = 0$;

(c) J_k is inadmissible in $||\phi(r_{k-1}+a_{ij})||$;

(d) $r_k \geq r_{k-1}+a$, where $a = \min_{i,j} |a_{ij}|$;

(e) $\phi(r_k) \neq \phi(r_{k-1})$.

<u>Proof</u>. By lemma 7, since if (a), (b), (c), and (d) are false the hypotheses of that lemma are met, and (e) follows.

The importance of this lemma lies in the fact that it sharply restricts the possibility of a nonterminating play in which $\{\phi(r_k)\}$ converges.

We need two more preparatory results before proceeding to the proof of the main theorem. Given an interior ϕ-strategy of player I, define the "punishment" function.

$$\pi(r, j) = \sum_i \xi_i(c)\phi(r+a_{ij}) - c,$$

where $c = \phi(r)$. (Compare proposition (B) above.) By lemma 6, π is always nonnegative.

LEMMA 9. If j is inadmissible in $||\phi(r+a_{ij})||$ then $\pi(r, j) > 0$.

<u>Proof</u>. By proposition B and the definition of interior ϕ-strategy we have

$$\sum_i \xi_i(c) \inf_{s \in C} \phi(s+a_{ij}) > c,$$

where $C = \phi^{-1}(c)$, $c = \phi(r)$. The required inequality is now obtained by removing the "inf" and substituting r for s.

LEMMA 10. If player I uses an interior ϕ-strategy against any strategy of player II, then with probability one the sequence $\{\pi_k\} = \{\pi(r_{k-1}, j_k)\}$ converges to 0.

Proof. (Compare [5], page 297, theorem 1.2 (1).) Consider the sum of the π_k. For each $n \geq 1$ we have:

$$E\left\{\sum_1^n \pi_k\right\} = E\left\{\sum_1^n \left[E\{\phi(r_k) \mid r_{k-1}\} - \phi(r_{k-1})\right]\right\}$$

$$= \sum_1^n \left[E\{\phi(r_k)\} - E\{\phi(r_{k-1})\}\right]$$

$$= E\{\phi(r_n)\} - E\{\phi(r_0)\}$$

$$\leq 1.$$

It follows that the probability of the infinite sum exceeding any given bound M is $\leq 1/M$. Hence, with probability one the series has a finite sum and $\{\pi_k\}$ converges to 0.

Proof of theorem 4. The theorem is easy if there is a positive row or negative column in $||a_{ij}||$. We therefore assume

$$\max_i \min_j a_{ij} < 0 < \min_j \max_i a_{ij}.$$

Let player I adopt an interior ϕ-strategy, and player II an arbitrary strategy. The play of the game that the occurs can be described by the pair of sequences $\{i_k\}$, $\{j_k\}$, which we shall regard as the underlying random variable. They are sufficient to determine three other important sequences $\{r_k\}$, $\{\pi_k\}$, and $\{\phi(r_k)\}$. The last-named is a bounded semimartingale, by lemma 0, and therefore the set of plays $\{i_k\}$, $\{j_k\}$ for which it fails to converge has probability zero. The set of plays for which

$\left\{ \pi_k \right\}$ fails to converge to 0 also has probability zero, by lemma 10. We shall prove that every play <u>outside</u> these two sets terminates. By (5) this will imply that player I's interior ϕ-strategy assures him an expected payoff $\geq \phi(r_0)$. The corresponding argument for the other player will complete the proof.

Consider therefore a play $\left\{ i_k \right\}$, $\left\{ j_k \right\}$ in which both

(9) $\qquad \left\{ \phi(r_k) \right\} \to c$

and

(10) $\qquad \left\{ \pi_k \right\} \to 0.$

If c = 0 and 1 then play must terminate, because of the fact that ϕ is discontinuous at 0 and R (see the corollary of lemma 3, section 2). Thus our object will be to show that the hypothesis $0 < c < 1$ leads to a contradiction. Let $\left\{ k_n \right\}$ be the sequence of indices k such that $\phi(r_k) \neq c$ or $\pi_k \neq 0$, or both, and let $s_n = r_{k_n}$. Lemma 8 shows that $\left\{ k_n \right\}$ and $\left\{ s_n \right\}$ have infinitely many terms, since alternatives "(a)" and "(b)" are excluded by hypothesis, while any unduly long chain of "(d)" will take r_k out of the interval $C = \phi^{-1}(c)$. Thus, instances of "(c)" or "(e)" must occur regularly, giving us $\pi_k \neq 0$, by lemma 9, or $\phi(r_k) \neq \phi(r_{k-1})$. In fact, consecutive terms of $\left\{ k_n \right\}$ cannot differ by more than $[(\gamma/a) + 2]$, where γ is the length of C, $a = \min|a_{ij}|$, and $[x]$ is the greatest integer $\leq x$.

Let us now examine the possible limit points of the sequence $\left\{ s_n \right\}$, bearing in mind properties (9) and (10). First we have

the upper and lower endpoints of C; call them u and ℓ respec-
tively. (If C is empty, a short argument shows that (9) is
contradicted.) Secondly, there are the points within C, in
the neighborhood of which π_k can be arbitrarily small, but
positive. There are only a finite number of such points, since
for each j, $\pi(r, j)$ is a monotonic increasing function of r in
C; denote these points by y_3, y_4, ..., y_p. Let δ be a small
positive constant (it has an exact value, which will be defined
later), and let Y_1, Y_2, ..., Y_p be a collection of intervals,
variously open and closed, but all of length δ, defined as
follows:

$$Y_1 = \begin{cases} [u, \ u+\delta) & \text{if } u \notin C \\ (u, \ u+\delta) & \text{if } u \in C \end{cases}$$

$$Y_2 = \begin{cases} (\ell-\delta, \ \ell] & \text{if } \ell \notin C \\ (\ell-\delta, \ \ell) & \text{if } \ell \in C \end{cases}$$

$$Y_1 = (y_1, \ y_1+\delta] \qquad\qquad 3 \leq 1 \leq p.$$

Let Y denote their set-theoretic union. By (9), (10) there is
an n_0 (depending on δ) such that $s_n \in Y$ for all $n > n_0$.

Now define δ to be the smallest nonzero number of the form:

$$(11) \qquad \left| \sum_{\nu=1}^{N} a_{i_\nu j_\nu} \right|, \qquad\qquad N \leq p[(\gamma/a)+2].$$

The effect of this definition is to ensure that when $|m-n| \leq p$,
the difference $|s_m - s_n|$ is either 0 or $\geq \delta$. It follows that,
for $n > n_0$, all s_n lying in a given Y_1 are equal (proof below).

Hence some s^* appears infinitely often in the sequence $\{s_n\}$.
If $s^* \notin C$ then (9) is contradicted. If $s^* \in C$ then (10) is
contradicted, since w_{k_n} is infinitely often positive, and
bounded away from 0 by the smallest nonzero $\pi(s^*, j)$. This is
the desired contradiction.

Finally, we have to prove the statement above in italics.
Call a pair (s_m, s_n) <u>conflicting</u> if they lie in the same Y_i
but are unequal. Either the statement in question is true or
there is a conflicting pair (s_m, s_n), coming after s_{n_0}, and
spanning no other conflicting pair $(s_{m'}, s_{n'})$, $m \leq m' < n' \leq n$.
Let $m_0 = m$ and let m_{j+1} be the last integer $\leq n$ such that $s_{m_{j+1}}$
and s_{m_j+1} lie in a common interval Y_i. Then the ascending
sequence

$$m_0, \ m_1, \ m_2, \ \ldots, \ m_{q-1}, \ m_q = n$$

has at most $p+1$ terms, and has the property that $s_{m_{j+1}} = s_{m_j+1}$
for each j. We see that each s_{m_j} is the sum of $s_{m_{j-1}}$ and at
most $[(\delta/a)+2]$ increments a_{i_ν, j_ν}. Hence $|s_m - s_n|$ is of the form
(11), and is either 0 or $\geq \delta$. This contradicts the assumption
that (s_m, s_n) was a conflicting pair.

5. EXISTENCE OF A VALUE IN GENERAL

When there are zeros in the $||a_{ij}||$ matrix a unique solution
to the fundamental equations (3), (4) is no longer assured, and
the value of the survival game may depend on the double-survival
payoff Q. In this section we show that the value does exist
if Q is sufficiently regular. Our proof makes use of a new

pair of functional equations involving the notion of recursive game (equations $(\bar{3})$, $(\bar{4})$ below), which reduce to (3), (4) if $||a_{1j}||$ is zero-free, and which always have a unique solution.

The restriction on Q consists in assuming that whenever $\{r_k\}$ converges to a limit r within (0, R), the payoff depends only on that limit: $Q = \bar{P}(r)$. We assume $\bar{P}$ to be monotonic increasing and, of course, bounded between O and 1. It is convenient to amalgamate it with the old function P, which was defined only outside (0, R), and denote both by $\bar{P}$. The still-arbitrary portion of Q will be denoted by $\bar{Q}$. Thus, the principal distinction in the present set-up is between convergent play (payoff $\bar{P}$) and nonconvergent play (payoff $\bar{Q}$), replacing the former distinction between terminating play (payoff P) and non-terminating play (payoff Q).

Note that the present arrangement still includes the important special cases Q ≡ constant. On the other hand, the mischievous Gale-Stewart functions (example 3 in section 1) are kept out.

THEOREM 5. If the convergent-play payoff is a monotonic function $\bar{P}$ of the limit of the first player's fortune, then the value of the survival game exists and is independent of the payoff $\bar{Q}$ for nonconvergent play.

The proof follows the same general lines as the proof of theorem 3 in section 5. However we must begin with some new definitions and notation.

By an <u>elementary recursive game</u> $[M, p]$ we shall mean the following: A matrix $||m_{ij}|| = M$ is given, each entry being either a number or the symbol $\textcircled{R}$. Players I and II choose i_1 and j_1 respectively. If $m_{i_1 j_1}$ is a number the game ends, and II pays I the indicated amount. If $m_{i_1 j_1}$ is $\widehat{R}$ there is no payment, and the players go back to the beginning and make new choices i_2 and j_2, etc. In case of infinite repetition the payoff is the number p (a constant).

The above is a special case of the recursive games defined by Everett [6], except for the "p" feature. (Everett effectively assumes that $p = 0$. However, $[M, p]$ is obviously equivalent to $[M', 0]$, with M' obtained by subtracting p from each numerical entry of M.) Translating the results of [6] we find that the elementary recursive game has a value, which we shall denote by $\overline{\text{val}}\ [M, p]$, though perhaps not optimal strategies. The value satisfies the relation

$$(12) \qquad x = \text{val}\ ||m_{ij}:x||,$$

where $||m_{ij}:x||$ is the matrix obtained by inserting the numerical variable x in place of $\widehat{R}$ in $||m_{ij}||$. The solutions of (12) form a closed interval, and it develops that $\overline{\text{val}}\ [M, p]$ is the solution of (12) that is closest to p.

The new functional equations can now be formulated; they are:

$$(3) \qquad \phi(r) = \overline{\text{val}}\ [M(\phi, r), P(r)], \qquad 0 < r < R$$

and

(4) $\qquad \phi(r) = \bar{F}(r), \qquad\qquad r \le 0, \ r \ge R,$

where $M(\phi, r)$ is the matrix with entries:

$$m_{ij}(\phi, r) = \begin{cases} \phi(r + a_{ij}) & \text{if } a_{ij} \ne 0 \\ \circledR & \text{if } a_{ij} = 0 \end{cases}$$

($\bar{4}$) and (4) are actually the same. If $||a_{ij}||$ is zero-free, then ($\bar{3}$) reduces to (3). In general, if ϕ is a solution of ($\bar{3}$), then (12) gives us:

$$\phi(r) = \text{val}||m_{ij}(\phi, r) : \phi(r)|| = \text{val}||\phi(r + a_{ij})||;$$

that is, ϕ is a solution of (3) as well. The converse is not true, however, since in fact ($\bar{3}$), ($\bar{4}$) always have a unique solution, while (3), (4) do not.

As in section 2 (lemma 1), we can construct a monotonic solution $\bar{v}_0$ to ($\bar{3}$), ($\bar{4}$) by iterating the transformation T:

(7) $\qquad T\phi(r) = \begin{cases} \overline{\text{val}}\left[M(\phi, r), \bar{F}(r)\right] & 0 < r < R \\ \phi(r) & r \le 0, \ r \ge R, \end{cases}$

applied to the same initial function $\phi_0(r)$ which is 0 for $r < R$ and 1 for $r \ge R$. It is easily shown that the functions $T^n \phi_0(r)$ are monotonic in r and form a bounded increasing sequence; the limit is the desired function $\bar{v}_0$. (It can be interpreted as the value function of the $\bar{Q} \ne 0$ game (compare theorem 1), but there is no point in establishing this fact now, in view of the stronger result that will be proved as theorem 7.)

As before, we introduce certain generalized payoff functions $\bar{F}^*(r)$; they will be assumed monotonic increasing in $(-A, R+A)$. The symbols (3^*), ($\bar{4}^*$), (7^*) will refer to equations (3), ($\bar{4}$), (7) with $\bar{F}$ replaced by $\bar{F}^*$. The next lemma corresponds to lemma 4

in section 3.

LEMMA 11. Suppose that $(3*)$, $(4*)$ have a strictly monotonic solution $\overline{\phi}*$. Then the value of the generalized survival game exists and is equal to $\overline{\phi}*(r_0)$.

Proof. Relative to a particular play of the generalized survival game, define $k_0 = 0$ and let k_{n+1} be the first k (if any) such that $r_k \neq r_{k_n}$. The subsequence $\{s_n\} = \{r_{k_n}\}$ is finite in length if and only if $\{r_k\}$ converges. We now describe a "local recursive ϵ-optimal $\overline{\phi}*$-strategy" for player I; it resembles our previous "local" strategies, but is based on elementary recursive games instead of matrix games. Choose a sequence of positive numbers ϵ_0, ϵ_1, ϵ_2, ... with sum ϵ. Let player I begin by playing an ϵ_0-optimal strategy of the elementary recursive game $\left[M(\overline{\phi}*, r_0), F*(r_0)\right]$. If and when that strategy runs out (after k_1 moves, in fact), let him continue with an ϵ_1-optimal strategy of $\left[M(\overline{\phi}*, s_1), F*(s_1)\right]$, and so on. In general, on his $(k_n+1)^{st}$ move, he will be commencing an ϵ_n-optimal strategy of $\left[M(\overline{\phi}*, s_n), F*(s_n)\right]$. We wish to show that such a strategy, played against an arbitrary strategy of player II, causes $\{r_k\}$ to converge with probability one.

Define the infinite sequence $\{x_n\}$ as follows:

$$x_n = \begin{cases} \overline{\phi}*(s_n) & \text{if } \{s_i\} \text{ is defined through } i = n, \\ F*(s_{n_0}) & \text{if } \{s_i\} \text{ stops at } i = n_0 < n. \end{cases}$$

Our construction ensures that, for $n = 1, 2, \ldots,$

$$E\left\{x_n \mid x_{n-1}, \ldots, x_0\right\} \geq x_{n-1} - \epsilon_{n-1} .$$

Therefore the sequence x_0, $x_1 + \xi_0$, $x_2 + \xi_0 + \xi_1$, $x_3 + \xi_0 + \xi_1 + \xi_2$, etc. is a bounded semimartingale. We conclude that $\{x_n\}$ converges with probability one, with

$$E\{x_\infty \mid x_0\} \geq x_0 - \epsilon.$$

However, $\{x_n\}$ cannot converge if $\{s_1\}$ does not stop at some s_{n_0}, since the s_1 oscillate and $\bar{\phi}*$ is strictly monotonic. Hence $x_\infty = \bar{P}*(s_{n_0}) = \bar{P}*(\lim r_k)$, and we have:

$$E\{\bar{P}*(\lim r_k)\} \geq \bar{\phi}*(r_0) - \epsilon.$$

The rest of the proof is obvious.

We now particularized $\bar{P}*(r)$ to be $\bar{P}(r) + \epsilon(r-R-A)$, where ϵ is a positive constant and $A = \max|a_{ij}|$. (Compare section 3.) Using the same initial function as before:

$$\bar{\phi}_0^*(r) = \phi_0^*(r) = \begin{cases} \epsilon(r-R-A) & \text{if } r < R \\ 1 + \epsilon(r-R-A) & \text{if } r \geq R, \end{cases}$$

we generate a sequence $\{\bar{\phi}_n^*\} = \{T*^n \phi_0^*\}$ by iterating the new transformation $T*$, given by (7*). The next lemma corresponds to lemma 5.

LEMMA 12. If $\text{val}\|a_{ij}\| \geq 0$, and if $\bar{v}_0$ is not continuous at R, then for sufficiently small ϵ the sequence $\{\bar{\phi}_n^*\}$ just defined converges to a strictly monotonic solution of (3*), (4*).

The proof is essentially the same as the proof of lemma 5. The substitution of elementary recursive games for matrix games causes trouble only at one spot: the proof of $\bar{\phi}_1^* \geq \bar{\phi}_0^*$. The

difficulty is resolved by an appeal to the following fact: $p \geq 0$ and $\mathrm{val}||m_{1j}:0|| \geq 0$ together imply that $\overline{\mathrm{val}}(M, p) \geq 0$.

LEMMA 13. If both v_0 and F have jumps at R, then so does $\overline{v}_0$.

Proof. We use the fact that $\overline{v}_0$ is a solution of (3), (4). As in the proof of lemma 11, we can find a "local recursive ε-optimal $\overline{v}_0$-strategy" for player I that ensures that the sequence $\{x_n\}$ (as defined there, but with F for F^* and $\overline{v}_0$ for ϕ^*) converges with probability 1, and that

$$B\{x_\infty \mid x_0\} \geq \overline{v}_0(r_0) - \varepsilon.$$

This holds for any strategy of player II; we shall consider a particular one. By lemma 3 the jump in v_0 means that $||a_{1j}||$ has a set of columns that meets each row in a subrow that contains a negative element, or is all zero. The same strategy for player II used in the proof lemma 3, part B, guarantees a probability $\geq s^{[-R/a]} = \delta > 0$ that $\{r_k\}$ will never increase. This means that with probability $\geq \delta$, $\{r_k\}$ will converge to a limit $r \leq r_0 < R$. Hence

$$B\{x_\infty \mid x_0\} \leq (1 - \delta) + \delta \cdot F(R-).$$

This bound is < 1 because of the jump in F at R, and is independent of r_0. Thus, letting $\varepsilon \to 0$ we find that $\overline{v}_0(r_0)$ is bounded away from 1 for $0 < r_0 < R$, as was to be shown.

Proof of theorem 5. There is no loss of generality in assuming $\mathrm{val}||a_{1j}|| \geq 0$. The theorem is trivial if v_0 is continuous at R, since then $v_0 = v_1$ by lemmas 2 and 3, so we can

assume that v_0 has a jump at R. Assume for the moment that $\bar{P}$
also has a jump at R. Then, applying lemmas 13, 12, and 11 in
that order, we find that the $\bar{P}*$ games all have values as $\varepsilon \to 0$.
The uniform convergence ensures that the original game also has
a value, and it is clear that this value, being the limit of the
$\bar{P}*$ values, is independent of $\bar{Q}$. On the other hand, if $\bar{P}$ is
continuous at R, then we can approximate it uniformly by a
sequence of discontinuous, monotonic functions. The preceding
argument applies to the latter, and passing to the limit completes
the proof.

COROLLARY. Equations (3), (4) have a unique
solution, assuming only that $\bar{P}$ is monotonic and
satisfies (4).

Proof. Let ϕ be any solution of (3), (4) and consider the
game determined by $\bar{P}$, $\bar{Q}$, with $\bar{Q} = 1$. A "local recursive ε-optimal
ϕ-strategy" for player I (see proof of lemma 11) will guarantee
him an expected payoff of at least $\phi(r_0) - \varepsilon$ in this game. Thus:

$$v(r_0) \geq \phi(r_0),$$

if v is its value function. But v is also the value function of
the game defined by $\bar{Q} = 0$, by theorem 5. By symmetry we have:

$$v(r_0) \leq \phi(r_0).$$

Thus ϕ is uniquely determined.

6. APPROXIMATIONS AND BOUNDS FOR THE VALUE FUNCTION

In this section we extend to games of survival some of the known results for random walks with absorbing barriers—i.e., the gambler's ruin problem (see [7], chapter 14). The random walk on (0, R) with each step determined by the fixed random variable ξ leads naturally to a functional equation, highly reminiscent of our fundamental equation (3):

(13) $$\phi(r) = E\left\{\phi(r+\xi)\right\}, \qquad 0 < r < R.$$

It is satisfied by several functions associated with the random walk; among them is the probability $p_R(r)$ that a particle starting at r will reach R before it reaches 0. This "absorption" probability is uniquely determined by (13) and the familiar boundary condition:

(4) $$\phi(r) = P(r), \qquad r \leq 0, \ r \geq R;$$

assuming that ξ is not identically 0.

If it happens that $E\left\{\xi\right\} = 0$, then (13) has among its solutions all linear functions $A + Br$. Applying the two conditions $p_R(0) = 0$ and $p_R(R) = 1$ we get $A = 0$ and $B = 1/R$, or

$$p_R(r) \approx \frac{r}{R}, \qquad 0 < r < R.$$

This is not exact because the particle will in general be absorbed beyond, not at, the barriers 0 and R. Taking this fact into account, we obtain rigorous estimates:

$$\frac{r}{R + \nu} \leq p_R(r) \leq \frac{r + \mu}{R + \mu}, \qquad 0 < r < R,$$

where μ and ν are such that always $-\mu \leq \xi \leq \nu$.

If on the other hand, $E\{\xi\} \neq 0$, then there will be a unique, nonzero λ_0 such that

(14) $\qquad E\left\{e^{\lambda_0 \xi}\right\} = 1,$

provided that ξ takes on both positive and negative values with positive probability (see [7], page 302, or [1], page 284). Then (13) has among its solutions all functions of the form $A + Be^{\lambda_0 r}$. As before, this leads to an approximation

$$p_R(r) \approx \frac{e^{\lambda_0 r} - 1}{e^{\lambda_0 R} - 1} \qquad\qquad 0 < r < R$$

and bounds:

$$\frac{e^{\lambda_0 r} - 1}{e^{\lambda_0(R+\nu)} - 1} \leq p_R(r) \leq \frac{e^{\lambda_0(r+\mu)} - 1}{e^{\lambda_0(R+\mu)} - 1} \quad , \quad 0 < r < R.$$

The linear case first discussed (with $E\{\xi\} = 0$) corresponds to $\lambda_0 = 0$. Actually, it is not an exceptional case; this becomes evident if we introduce the function f:

$$f(\lambda, x) = \begin{cases} (e^{\lambda x} - 1)/\lambda & \text{if } \lambda \neq 0 \\ x & \text{if } \lambda = 0, \end{cases}$$

which is continuous and monotonic increasing in λ for each x. The relation (14) defining λ_0 becomes $E\{f(\lambda_0, \xi)\} = 0$, which has a unique solution in all cases. Write $F(x)$ for $f(\lambda_0, x)$. Then the approximation and bounds for p_R are simply $F(r)/F(R)$, $F(r)/F(R+\nu)$ and $F(r+\mu)/F(R+\mu)$, respectively, regardless of whether λ_0 is positive, negative, or zero.

For games of survival we have some very similar results:

LEMMA 14. If $||a_{ij}||$ is zero-free and if

$$\max_i \min_j a_{ij} < 0 < \min_j \max_i a_{ij},$$

then there is a unique number λ_0 such that

(15) $\text{val}||f(\lambda_0, a_{ij})|| = 0.$

Moreover, λ_0 and $\text{val}||a_{ij}||$ have opposite signs, or are both zero.

Proof. The value of $||f(\lambda, a_{ij})||$ is a continuous and strictly monotonic function of λ, since none of the a_{ij} is 0. This function tends to the limit $+\infty$ as $\lambda \to +\infty$ because of the positive element in each column, and to the limit $-\infty$ as $\lambda \to -\infty$ because of the negative element in each row. Therefore, it has precisely one zero. The last part of the lemma follows from the fact that $\text{val}||f(0, a_{ij})|| = \text{val}||a_{ij}||$.

Again write $F(x)$ for $f(\lambda_0, x)$.

LEMMA 15. F is a solution of (3).

Proof. Using the identity (valid for all λ):

$$f(\lambda, x+y) = f(\lambda, x) + e^{\lambda x} f(\lambda, y),$$

we have:

$$\text{val}||F(r+a_{ij})|| = F(r) + e^{\lambda_0 r} \text{val}||F(a_{ij})||$$

$$= F(r)$$

as required. The corollary which follows is proved by the same device.

COROLLARY. The local P-strategies are precisely those mixed strategies that use only probability distributions that are optimal in the matrix game $||P(a_{ij})||$.

THEOREM 6. If $||a_{ij}||$ is zero-free and if

$$\max_{i} \min_{j} a_{ij} < 0 < \min_{j} \max_{i} a_{ij}$$

then the value of the survival game is approximately equal to $P(r_0)/P(R)$. More precisely, we have:

(16)
$$\frac{P(r_0)}{P(R + \nu)} \leq v(r_0) \leq \frac{P(r_0 + \mu)}{P(R + \mu)} \quad ,$$

where $\nu = \max a_{ij}$, $\mu = -\min a_{ij}$. The local P-strategies are approximately optimal, in the sense that player I can enforce the lower bound of (16), and player II the upper bound, by using them.

We remark that an all-positive row ($\max \min a_{ij} > 0$) or an all-negative column ($\min \max a_{ij} < 0$) trivializes the game. (These cases correspond to $\lambda_0 = -\infty$ and $+\infty$ respectively.)

Proof of theorem 6. Denote the indicated lower bound in (16) by $g(r_0)$. Clearly g is a strictly monotonic solution of (3), and the local P-strategies are also local g-strategies. If we set $P^*(r) = g(r)$ for r outside (0, R) we have a "generalized survival game" in the sense of section 3. By lemma 4, $g(r_0)$ is

its value, and the local P-strategies are optimal. But P* $\leq$ P throughout the relevant intervals $(-\mu, 0]$ and $[R, R+\nu)$; therefore $\underline{g}(r_0) \leq v(r_0)$, and the local P-strategies enforce at least the lower amount for player I. The other bound is established in the same way.

The bounds (10) can sometimes be improved by exploiting special properties of the matrix. For example, inadmissible rows or columns of $||P(a_{ij})||$ can be disregarded in calculating μ and ν. Two other such results are the following:

COROLLARY 1. If r_0, R, and the a_{ij} are all integers, then μ and ν in (10) may be replaced by $\mu-1$ and $\nu-1$ respectively.

COROLLARY 2. If r_0 and R are integers and the a_{ij} are all ± 1, then the value of the game is exactly $P(r_0)/P(R)$, and the local R-strategies are optimal.

An equally exact result holds for arbitrary r_0 and R; it has the form $v(r_0) = P(r_0+ \bar\mu)/P(R+ \bar\mu + \bar\nu)$, where $-\bar\mu$ and $R+\bar\nu$ are the unique absorption points of the process. However, $\bar\mu$ and $\bar\nu$ depend on r_0 in such a way that v is actually a step function, despite the continuity of P.

Two simple asymptotic results are of interest:

COROLLARY 3. If $R \to \infty$ with r_0 held fixed, the value of the game tends to a limit ℓ that is = 0 or satisfies

$$0 < 1 - e^{\lambda_0 r_0} \leq \ell \leq 1 - e^{\lambda_0(r_0 + \mu)},$$

depending on whether $\text{val}||a_{ij}|| \leq 0$ or > 0.

Thus, if the "money" game is in his favor, player I can defeat even an arbitrarily rich opponent, with some probability.

COROLLARY 4. If r_0 and $R \to \infty$ in a fixed ratio, or, equivalently, if the a_{ij} all $\to 0$ in a fixed ratio, then the limit of $v(r_0)$ is either 0, r_0/R, or 1, depending on whether $\text{val}||a_{ij}|| < 0$, = 0, or > 0, respectively.

As we pass to the limit in this fashion, the "naive" strategy of maximizing the minimum expected money gain on each round becomes better and better. Indeed, in the balanced case ($\text{val}||a_{ij}|| = 0$) it is a local P-strategy, and in the lopsided cases ($\text{val}||a_{ij}|| < 0$ or > 0) one player has nothing to lose anyway, in the limit, while any strategy with positive expected gain wins for the other. These remarks may clarify the rather puzzling (and not entirely correct) conclusions of [1], [2], [11] to the effect that the "naive" strategy just mentioned is approximately optimal.

[The following is an example of a game in which the "naive" strategy is not satisfactory. In fact, if player I follows it

$$\begin{pmatrix} -\varepsilon & -\varepsilon \\ -10 & 9 \\ 9 & -10 \end{pmatrix}$$

Example 10

here he will always choose the first row and hence always
lose (assuming $0 < \varepsilon < 1/2$). Another case where the "naive"
strategy is not satisfactory for player I is given by example
7 above.]

A generalization of corollary 4 has been obtained by Scarf
[12] for survival games in which r and a_{ij} are n-dimensional
vectors. Under certain assumptions, which reduce to our condi-
tion $val||a_{ij}|| = 0$, he finds that the limiting value functions
are generalized harmonic functions, being the zeros of certain
second-order differential operators, in general nonlinear.

A different extension of the survival game model, of some
interest, is obtained by changing the information pattern, dis-
rupting in some specified way the process whereby the players
learn of each other's past moves and the resulting winnings or
losses (see [14]). Since the local F-strategies can be played
without benefit of any information whatever, the bounds of
theorem 6 remain applicable, and we have:

COROLLARY 5. In a game of survival with restricted
information flow, the value (if it exists) lies within
the bounds (15). In any case, the minorant (sup-inf) and
majorant (inf-sup) values exist and satisfy (16).

We note in passing that the value always exists if $Q \equiv 0$ or
$Q \equiv 1$, since the payoff as a function of the pure strategies is
semicontinuous, and the pure strategy spaces are compact, regard-
less of the information pattern (compare [13]).

So far in this section we have been proceeding on the assump-
tion that $||a_{ij}||$ is zero-free. We now indicate without proof

the modifications required if this assumption is dropped. The parallel numbering will assist comparison.

LEMMA 14°. If $\max\limits_{i} \min\limits_{j} a_{ij} < 0 < \min\limits_{j} \max\limits_{i} a_{ij}$ then the solutions of:

(15°) $\text{val}||f(\lambda, a_{ij})|| = 0$

constitute a finite, closed interval $[\lambda', \lambda'']$. Moreover, λ' and λ'' both have signs opposite to $\text{val}||a_{ij}||$, and we have $\lambda' \leq 0 \leq \lambda''$ if and only if $\text{val}||a_{ij}|| = 0$.

Write $F'(x)$ for $f(\lambda', x)$ and $F''(x)$ for $f(\lambda'', x)$.

LEMMA 15°. Both F' and F'' are solutions of (3).

COROLLARY. The local F'- and F''-strategies are precisely those mixed strategies that use only probability distributions that are optimal in the matrix games $||F'(a_{ij})||$ and $||F''(a_{ij})||$ respectively.

THEOREM 6°. If $\max\limits_{i} \min\limits_{j} a_{ij} < 0 < \min\limits_{j} \max\limits_{i} a_{ij}$ then the extreme solutions v_0 and v_1 of (3), (4) are approximated by $F''(r)/F''(R)$ and $F'(r)/F'(R)$ respectively, with precise bounds of the form (16). In the $Q = 0$ game, player II can enforce the upper bound to v_0 by playing a local F''-strategy, and player I can enforce to within any $\varepsilon > 0$ of the lower bound by choosing $\delta > 0$ small enough and playing optimal strategies of $||f(\lambda''+\delta, a_{ij})||$ on each round. A similar statement holds for the $Q = 1$ game and its value function v_1. For general Q the value (if it exists) lies between $F''(r_0)/F''(R+\nu)$ and $F'(r+\mu)/F'(R+\mu)$.

Again we remark that the cases max min $a_{ij} > 0$ and min max $a_{ij} <$
                      i j   j i
are trivial. A guide to what happens when one or both is equal
to zero is provided by lemma 5, in section 2.

The five corollaries are unchanged or are modified in the
obvious way, using the last part of lemma 14^0 and noting that
statements must be made in terms of v_0 and v_1, with the value of
the game in general (if it exists) lying in between. Corollary
2^0 can be extended slightly (with the aid of theorem 2) to yield
the following result:

THEOREM 7. If the a_{ij} are all ± 1 or 0, and if
$\lambda' = \lambda''$, then $v_0 = v_1$ and the value of the survival
game exists and is independent of Q.

It is natural to ask whether $\lambda' = \lambda''$ implies $v_0 = v_1$ under
more general conditions. In view of example 11, discussed below,
the answer seems to be in the negative. However, the converse
implication is valid almost always. In fact, if $\lambda' < \lambda''$ then
the inequality:

$$\frac{F''(r+\mu)}{F''(R+\mu)} < \frac{F'(r)}{F'(R+\nu)}$$

holds at $r = R/2$ if R is sufficiently large. This implies
$v_0(R/2) < v_1(R/2)$ by theorem 6^0. This proves:

THEOREM 8. If $||a_{ij}||$ is such that the functions
v_0 and v_1 are identical for large values of R, then
$\lambda' = \lambda''$.

(To see that the condition on R is needed, go back to example 2

< 0

in section 1 and put R = 1. Then we have $v_0 = v_1 = 1/2$ in $(0, R)$;
however $\lambda' < 0 < \lambda''$.)

[In our final example, it is easily checked that
$val||a_{ij}|| = 0$ and that $\lambda' = \lambda'' = 0$. To show that v_0
and v_1 are different, let u' and u" be the value functions
for the survival games which correspond to the submatrices:

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$$

respectively. Clearly, $v_1 \geq \max[u', u'']$ and $v_0 \leq \min[u', u'']$.
However, a simple calculation shows that u' and u" are dis-
tinct for $R > 1$; hence v_0 and v_1 are also distinct.]

$$\begin{pmatrix} 0 & 0 & 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 \\ 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \end{pmatrix}$$

Example 11

[1]. Bellman, R., On multi-stage games with imprecise payoff, RAND Corporation Research Memorandum RM-1337, September 1954.

[2]. ————, Decision-making in the face of uncertainty —II, Naval Research Logistics Quarterly 1 (1954), 327-332.

[3]. ———— and LaSalle, J., On non-zero-sum games and stochastic processes, RAND Corporation Research Memorandum RM-212, August 1959.

[4]. Blackwell, D. On multi-component attrition games, Naval Research Logistics Quarterly 1, (1954), 210-216.

[5]. Doob, J. L., Stochastic Processes, Wiley, 1953.

[6]. Everett, H., Recursive Games, this study.

[7]. Feller, W., Probability Theory and its Applications, Wiley, 1950.

[8]. Gale, D. and Stewart, F. M., Infinite games with perfect information, Annals of Mathematics Study No. 28 (1953), 245-266.

[9]. Glicksberg, I., Minimax theorem for upper and lower semi-continuous payoffs, RAND Corporation Research Memorandum RM-478, October 1950.

[10]. Hausner, M., Games of survival and Optimal strategies in games of survival, RAND Corporation Research Memoranda RM-776 and RM-777, February 1952.

[11]. Peisakoff, M. P., More on games of survival, RAND Corporation Research Memorandum RM-884, June 1952; reproduced in R. Bellman: The Theory of Dynamic Programming, RAND Corporation Report R-245 (1953), 87-95.

[12]. Scarf, H. E., On differential games with survival payoff, this study.

[13]. ———— and Shapley, L. S., Games with information lag, RAND Corporation Research Memorandum RM-1320, August 1954.

[14]. ———— and Shapley, L. S., Games with partial information, this study.

[15]. Wald, A., On cumulative sums of random variables, Annals of Math. Stat. 15 (1944), 283-296.

NOTE: The phrase "this study" in the above refers to the forthcoming Annals of Mathematics Study No. 39: Contributions to the Theory of Games, Vol. III, edited by M. Dresher and A. W. Tucker, to which the present paper is being submitted.